MATHEMATICS

Chapter 14: MATHEMATICAL REASONING

MATHEMATICAL REASONING

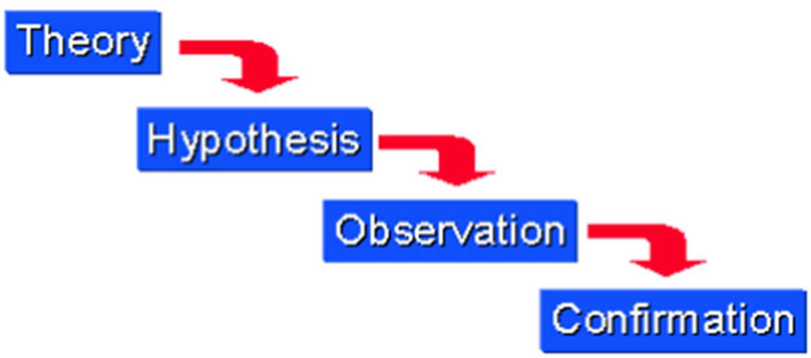
Key Concepts

1. There are two types of reasoning—deductive and inductive. Deductive reasoning was developed by Aristotle, Thales and Pythagoras in the classical period (600 to 300 BC).

2. In deductive reasoning, given a statement to be proven, often called a conjecture or a theorem, valid deductive steps are derived and a proof may or may not be established. Deductive reasoning is the application of a general case to a particular case.

3. Inductive reasoning depends on the working with each case and developing a conjecture by observing incidence till each and every case is observed.

4. Deductive approach is known as the 'top–down' approach. Given is the theorem which is narrowed down to specific hypotheses and then to observation. Finally, the hypothesis is tested with specific data to get the confirmation (or not) of the original theory.



5. Mathematical reasoning is based on deductive reasoning.

   The classic example of deductive reasoning, given by Aristotle, is:

   - All men are mortal.
   - Socrates is a man.
   - Socrates is mortal.

6. The basic unit involved in reasoning is a mathematical statement.

7. A sentence is called a mathematically acceptable statement if it is either true or false but not both. A sentence which is both true and false simultaneously is called a paradox.

8. Sentences which involve tomorrow, yesterday, here, there etc., i.e. variables etc., are not statements.

9. The sentence which expresses a request, command or question is not a statement.

10. The denial of a statement is called the negation of the statement

11. Two or more statements joined by words such as ‘and’ and ‘or’ are called compound statements. Each statement is called a component statement. The words ‘and’ and ‘or’ are connecting words.

12. An ‘And’ statement is true if each of the component statements are true, and it is false even if one of the component statements is false.

13. An ‘OR’ statement will be true when even one of its components is true and is false only when all of its components are false.

14. The word ‘OR’ can be used in two ways—inclusive OR and exclusive OR. If only one of the two options is possible, then the OR used is exclusive OR.

    If any one of the two options or both the options are possible, then the OR used is an inclusive OR.

15. There exists ‘$\exists$’ and ‘For all’ $\forall$ are called quantifiers.

16. A statement with quantifier ‘There exists’ is true if it is true for at least one case.

17. If p and q are two statements, then a statement of the form ‘If p then q’ is known as a conditional statement. In symbolic form, p implies q is denoted by $p \Rightarrow q$.

18. The conditional statement $p \Rightarrow q$ can be expressed in various other forms

    (i) q if p (ii) p only if q (iii) p is sufficient for q (iv) q is necessary for p.

19. A statement formed by the combination of two statements of the form if p then q and if q then p and is called if and only if implication. Also, it is denoted by $p \Leftrightarrow q$. It is called a biconditional statement.

20. Contra positive and converse can be obtained by a ‘if then’ statement. The contrapositive of a statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$.

    The converse of a statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.

21. **Truth values of various statement**

p	q	p and q	p or q	p⇒q
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

22. To prove the truth of a 'if p then q' statement, there are two ways. The first is to assume p is true and prove q is true. This is called the direct method.

Or

Assume that q is false and prove p is false. This is called the contrapositive method.

23. To prove the truth of 'p if and only if q' statement, we must prove two things—one that the truth of p implies the truth of q and the second is that the truth of q implies the truth of p.

24. The following methods are used to check the validity of statements:

i. Direct method.
ii. Contra positive method.
iii. Method of contradiction.
iv. Using a counter example.

25. To check whether a statement p is true, we assume that it is not true, i.e. ~p is true. Then we arrive at some result which contradicts our assumption.

MIND MAP : LEARNING MADE SIMPLE CHAPTER - 14

Mathematical Reasoning

Quantifiers

In this statement, the two important symbols are used.

- The symbol '∀' stand for "all values of";
- The symbol '∃' stand for "there exists"

Eg: For every prime number P, P is an irrational number.

- This means that if S denotes the set of all prime numbers, then for all the members of P of the set S, P is an irrational number.

Compound Statement

Many mathematical statements are obtained by combining one or more statements using some connecting words like "and", "or" etc. Each statement is called a "Compound Statement."

Eg:

'The sky is blue' and 'the grass is green' is a compound statement where connecting word is "and";
the components of Compound Statement are

p : The sky is blue

q : The grass is green

Sentence

A sentence is called a mathematically acceptable statement if it is either true or false but not both.

Eg:

- The sum of two positive numbers is positive.
- All prime numbers are odd numbers.
- In above statements, first is 'true' and second is 'false'.

Negation

A statement which is formed by changing the true value of a given statement by using the word like 'no', 'not' is called negation of given statement.

- If P is a statement, then negation is denoted by $\sim p$

Eg: New Delhi is a city.

The negation of this statement:

→ It is not the case that New Delhi is a city.

→ It is false that New Delhi is a city.

→ New Delhi is not a city.

Contrapositive and Converse Statements

- The contrapositive of a statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$
- The converse of a statement $p \Rightarrow q$ is the statement $q \Rightarrow p$

Eg: If the physical environment changes, then biological environmental changes.

- **Contrapositive :** If the biological environment does not change then the physical environment does not change.
- **Converse:** If the biological environment changes then physical environment changes.

Validity of Statements

The following methods are used to check the validity of statements

(i) Direct Method

(ii) Contrapositive Method

(iii) Method of Contradiction

(iv) Using a Counter example

Implications

- These are statements with word "if then", "only if" and "if and only if".

Eg:

- r: if a number is a multiple of 9, then it is multiple of 3.
- p: a number is a multiple of 9.
- q: a number is a multiple of 3.
- Then, if p then q is the same as following:

→ p implies q is denoted by $p \Rightarrow q$, then symbol $\Rightarrow$ stands for implies,

→ p is a sufficient condition for q. then symbol $\Rightarrow$

→ p only if q

→ q is a necessary condition for p

→ $\sim q$ implies $\sim p$

Important Questions

Multiple Choice questions-

Question 1. Which of the following statement is a conjunction?

(a) Ram and Shyam are friends

(b) Both Ram and Shyam are friends

(c) Both Ram and Shyam are enemies

(d) None of these

Question 2. Which of the following is true?

(a) A prime number is either even or odd

(b) √3 is irrational number.

(c) 24 is a multiple of 2, 4 and 8

(d) Everyone in India speaks Hindi.

Question 3. The contrapositive of the statement If a number is not divisible by 3, it is not divisible by 9 is:

(a) If a number is divisible by 3, it is not divisible by 9

(b) If a number is not divisible by 3, it is divisible by 9

(c) If a number is divisible by 3, it is divisible by 9

(d) If a number is not divisible by 3, it is not divisible by 9

Question 4. The negation of the statement The product of 3 and 4 is 9 is

(a) It is false that the product of 3 and 4 is 9

(b) The product of 3 and 4 is 12

(c) The product of 3 and 4 is not 12

(d) It is false that the product of 3 and 4 is not 9

Question 5. The connective in the statement Earth revolves round the Sun and Moon is a satellite of earth is

(a) or

(b) Earth

(c) Sun

(d) and

Question 6. If p is a statement then the negation of p is

(a) p

(b) -p

(c) ~p

(d) p~

Question 7. If (p or q) is false when

(a) p is true and q is false

(b) p is true and q is true

(c) p is false and q is false

(d) p is false and q is true

Question 8. If (p and q) is true then

(a) p is true and q is false

(b) p is false and q is false

(c) p is false and q is true

(d) p is true and q is true

Question 9. If (p and q) is false then

(a) p is true and q is false

(b) p is false and q is false

(c) p is false and q is true

(d) all of the above

Question 10. The compound statement with AND is true if all its component statements are

(a) true

(b) false

(c) either true or false

(d) None of these

Short Questions:

1. Give three examples of sentences which are not statements. Give reasons for the answers.
2. Write the negation of the following statements

   (i) Chennai is the capital of Tamil Nadu.

   (ii) Every natural number is an integer.
3. Find the component statements of the following compound statements and check whether they are true or false.

(i) The number 3 is prime or it is odd.

4. Check whether the following pair of statements are negations of each other Give reasons for your answer.

(i) x + y = y + x is true for every real numbers x and y.

(ii) There exists real numbers x and y for which x + y = y + x.

5. Write the contra-positive and converse of the following statements.

(i) If x is a prime number, then x is odd.

(ii) if the two lines are parallel, then they do not intersect in the same plane.

6. Show that the statement

P : "If x is a real number such that x3 + 4x = 0 then x is 0" is true by

(i) direct method, (ii) method of contradiction, (iii) method of contra-positive.

7. Given below are two statements

P : 25 is a multiple of 5.

q: 25 is a multiple of 8

Write the compound statements connecting these two statements with "and" and "OR". In both cases check the validity of the compound statement.

8. Write the following statement in five different ways, conveying the same meaning.

P : If a triangle is equiangular, then it is an obtuse angled triangle.

Answer Key:

MCQ:

1. (d) None of these

2. (d) Everyone in India speaks Hindi.

3. (c) If a number is divisible by 3, it is divisible by 9

4. (a) It is false that the product of 3 and 4 is 9

5. (d) and

6. (c) ~p

7. (a) p is true and q is false

8. (d) p is true and q is true

9. (d) all of the above

10.(a) true

Short Answer:

1. (i) The sentence "Rani is a beautiful girl" is not a statement. To some Rani may look beautiful and to other she may not look beautiful. We cannot say on logic whether or not this sentence is true.

   (ii) The sentence 'shut the door' is not a statement. It is only an imperative sentence giving a direction to someone. There is no question of it being true or false.

   (iii) The sentence 'yesterday was Friday' is not a statement. It is an ambiguous sentence which is true if spoken on Saturday and false if spoken on other days. Truth or false hood of the sentence depends on the time at which it is spoken and not on mathematical reasoning.

2. (i) Chennai is not the capital of Tamil Nadu.

   (ii) Every natural number is not an integer.

3. The component statements of the given statement are

   p: "The number 3 is prime"

   q: "number 3 is odd"

   There two have been connected by using the connective "or"

   The given statement is true as both the statements are true.

4. The given statements are

   P : " x + y = y + x is true for every real number x and y

   q : "There exists real numbers x and y for which x + y = y + x".

   These statements are not negations of each other as they can be true at the same time. Infect, negation of p is

   $\sim p$ " There are real numbers x and y for which x + y ≠ y + x."

   Note that p is always true whatever and may be and is always false.

5. If statement is p ⇒ q then its contra-positive is $\sim q \;\Rightarrow \sim\; p$ and its converse is q ⇒ p.

   (i) Contra-positive : "If x is not odd, then is not a prime number."

   Converse : "If x is odd, then x is a prime number."

   (ii) Contra-positive : "If two lines intersect in the same plane, then they are not parallel."

   Converse: "If two lines do not intersect in the same plane, then they are parallel."

6. Given statement is p: "If x is a real number such that $x^3 + 4x = 0$, then x = 0"

   (i) Direct method: Let x3 + 4x = 0, x ϵ R

   $\Rightarrow x(x^2+4)=0, x\in R \Rightarrow x=0\ (\because \text{if } x\in R \text{ then } x^2+4\geq 4)$

   Note that if the product of two numbers is zero then atleast one of them is surely zero.

   Thus, we find that p is a true statement.

   (ii) Method of contradiction.

Let x be a nonzero real number

$\Rightarrow x^2 > 0$ (∵Square of a non- zero real number is always positive)

$\Rightarrow x^2 + 4 > 4 \Rightarrow x^2 + 4 \neq 0$

$\Rightarrow x(x^2 + 4) \neq 0$ $(\because x \neq 0 \text{ and } x^2 + 4 \neq 0)$

$\Rightarrow x^3 + 4x \neq 0$, which is a contradiction.

Hence, $x = 0$

(iii) Method of contra-positive:

Let $q: "x \in R \text{ and } x^3 + 4x = 0"$

$r: "x = 0"$

∴Given statement p is $q \Rightarrow r$

Its contra-positive is $\sim r \Rightarrow \sim q$

i.e. "if x is a non- zero real number then $x^3 + 4$ is also nonzero"

Now $x \neq 0, x \in R \Rightarrow x^2 > 0 \Rightarrow x^2 + 4 > 4 \Rightarrow x^2 + 4 \neq 0$

$\Rightarrow x(x^2 + 4) \neq 0 \Rightarrow x^3 + 4x \neq 0$ $i.e. \sim r \Rightarrow \sim q.$

Thus the statement is $\sim r \Rightarrow \sim q.$ always true

Hence, $q \Rightarrow r$ is always true

Note: Infact, 'Method of contradiction' is another form of 'contra-positive method' while proving an implication.

7. Case I. Using the connective 'and', we obtain the compound statement "p and q".

i.e., "25 is a multiple of 5 and 8".

It is false statement as q is always false. (∵ 25 is not multiple of 8)

Case II. Using the connective 'or', we obtain the compound statement "p or q"

i.e., "25 is a multiple of 5 or 8".

It is a true statement as p always true. (∵ 25 is a multiple of 5)

8. Given statement is

"If a triangle is equiangular, then it is an obtuse angled triangle". Its five equivalents are as follows:

(i) "A triangle is equiangular only if it is an obtuse angled triangle".

(ii) "If a triangle is not obtuse angled triangle then it is not an equiangular triangle."

(iii) "equiangularity is a sufficient condition for triangle to be obtuse angled."

(iv) "A triangle being obtuse angled, is necessary condition for it to be equiangular".

(v) A triangle is obtuse is obtuse angled if it is equiangular.